

# A generalization of Kronecker's first limit formula

Amod Agashe\*

## Abstract

Kronecker's first limit formula gives the polar and constant terms of the Laurent series expansion of the Eisenstein series for  $\mathrm{SL}(2, \mathbf{Z})$  at  $s = 1$ . In this article, we generalize the formula to certain maximal parabolic Eisenstein series associated to  $\mathrm{SL}(n, \mathbf{Z})$  for  $n \geq 2$ . We also indicate how the generalized formula can be used to give the polar and constant terms of the zeta function of any number field at  $s = 1$ .

## 1 Introduction and results

Let  $n \geq 2$  be an integer. Let  $\tau$  be in the generalized upper half-plane  $\mathfrak{H}^n$ , which consists of  $n \times n$  matrices with real number entries that are the product of an upper triangular matrix with 1's along the diagonal and a diagonal matrix with positive diagonal entries such that the lowermost diagonal entry is 1. When  $n = 2$ , one can identify  $\mathfrak{H}^n$  with the usual complex upper half plane. For details, see, e.g., [Gol06, §1.2].

In the following,  $m_1, \dots, m_n$  denote integers with perhaps some added restrictions as noted; in particular, we follow the convention that in any sum over a subset of  $m_1, \dots, m_n$ , if a term has denominator zero for some values of  $m_1, \dots, m_n$ , then the term is to be skipped in the sum.

Consider the maximal parabolic Eisenstein series

$$E_n(\tau, s) = \sum_{(m_1, \dots, m_n)=1} \frac{(\det \tau)^s}{\| (m_1 \dots m_n) \tau \|^{ns/2}},$$

where  $\| (m_1 \dots m_n) \tau \|$  denotes the norm of the row vector that is the product of the row vector  $(m_1 \dots m_n)$  and the matrix  $\tau$ .

Let

$$E'_n(\tau, s) = \zeta(ns) E(\tau, s) = \sum_{m_1, \dots, m_n} \frac{(\det \tau)^s}{\| (m_1 \dots m_n) \tau \|^{ns/2}}.$$

Note that in the case where  $n = 2$ , if  $\tau$  corresponds to the point  $z = x + iy$  in the complex upper half plane, then

$$E'_2(\tau, s) = \sum_{m_1, m_2} \frac{y^s}{|m_1 z + m_2|^{2s}},$$

which is often denoted by the same symbol without the prime superscript (see, e.g., [Lan87, §20.4]). The classical Kronecker's first limit formula gives the first two terms of the Laurent expansion of  $E'_2(\tau, s)$  at  $s = 1$  (see, e.g., [Lan87, §20.4]):

$$E'_2(\tau, s) = \pi \left( \frac{1}{s-1} + (2\gamma - \log 4 - \log y - 4 \log |\eta(z)|) + O(s-1) \right), \quad (1)$$

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where  $\eta(z)$  is the Dedekind eta-function. Note that sometimes Kronecker's first limit formula is stated for Dedekind zeta functions or for Epstein zeta functions, but such formulas can be deduced from the formula above.

Kronecker's first limit formula was generalized to the case  $n = 3$  in [BG84] and [Efr92]. In this article, we generalize the formula to arbitrary  $n \geq 2$  (see Theorem 1.1). A generalization of Kronecker's first limit formula to arbitrary  $n \geq 2$  is also given in [Ter73] for Epstein zeta functions.

We also consider the function

$$\begin{aligned} E_n^*(\tau, s) &= \pi^{-ns/2} \Gamma(ns/2) \zeta(ns) E(\tau, s) \\ &= \pi^{-ns/2} \Gamma(ns/2) \sum_{m_1, \dots, m_n} \frac{(\det \tau)^s}{\| (m_1 \dots m_n)_\tau \|^{ns/2}}. \end{aligned} \quad (2)$$

Let  $y_1, \dots, y_{n-1}$  denote the unique positive real numbers such that for  $i \geq 1$ , we have  $\tau_{n-i, n-i} = \prod_{j=1}^i y_j$ .

**Theorem 1.1.** *If  $m_1, \dots, m_n$  are integers, then for  $j = 1, \dots, n$ , let  $b_j = \sum_{i=1, \dots, j-1} m_i \tau_{i,j}$  and  $c_j = \tau_{j,j}$ ; also let  $m$  be the nonnegative real number such that  $m^2 = m_n^2/c_n^2 + m_{n-1}^2/c_{n-1}^2 + \dots + m_2^2/c_2^2$  and  $d = b_n m_n/c_n + b_{n-1} m_{n-1}/c_{n-1} + \dots + b_2 m_2/c_2$ . Let  $\tau'$  be  $\tau$  with topmost row and leftmost column removed and let*

$$g(\tau) = \exp \left\{ -\frac{1}{4} \left( \left( \prod_{i=1}^{n-1} y_i^{\binom{i}{n-1}} \right) E_{n-1}^* \left( \tau', \frac{n}{n-1} \right) + \sum_{m_1 \neq 0} \frac{1}{|m_1|} \sum_{\substack{(m_2, \dots, m_n) \\ \neq (0, \dots, 0)}} \exp \left( 2\pi i d - 2\pi |m_1| m \prod_{i=1}^{n-1} y_i \right) \right) \right\}.$$

Then

$$\begin{aligned} E_n^*(\tau, s) &= \frac{2/n}{s-1} + \left( \gamma - \log 4\pi - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i^i \right) - 4 \log g(\tau) \right) + O(s-1), \text{ and} \\ E_n'(\tau, s) &= \pi \left( \frac{2/n}{s-1} + \left( 2\gamma - \log 4 - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i^i \right) - 4 \log g(\tau) \right) + O(s-1) \right). \end{aligned}$$

The theorem is proved in Section 2. The methods used are very elementary: one only needs some basic Calculus and the Poisson summation formula, which is recalled in equation (10). Our proof is a generalization of the proof of the classical Kronecker's first limit formula given in [Lan87, §20.4], and the key observation is to perform the sum in (2) over  $m_1, \dots, m_n$  conveniently (we first sum over  $\{m_1\}$  and then over  $\{m_2, \dots, m_n\}$ ; when  $n = 2$ , there is not much of a choice) and to apply the Poisson summation formula in the correct order (over  $m_n$  first, followed by  $m_{n-1}$ , and so on, up to  $m_2$ ). Even in the case  $n = 3$ , our proof differs in some key steps with that in [Efr92] (who introduces complex coordinates on  $\mathfrak{H}^3$ , while we don't) and that in [BG84] (who use minimal parabolic Eisenstein series). Our proof techniques are similar to those used in [Ter73] (about which we learned only after a first draft of this article was written), but are more direct and elementary (the main goal of [Ter73] is to prove the functional equation of the Epstein zeta function using generalizations of the Selberg-Chowla formula).

If we put  $n = 2$  in the formula for  $E_n'(\tau, s)$  above, then we get the classical Kronecker's first limit formula (1), with  $g(\tau) = |\eta(\tau)|$  (this last equality follows from the correctness of our formula, but also because our proof is identical to the classical proof in the case  $n = 2$  as given in [Lan87, §20.4]). Thus our function  $g(\tau)$  for arbitrary  $n$  is a generalization of  $|\eta(\tau)|$ . If we put  $n = 3$ , then we recover the formula for  $E_3^*(\tau, s)$  given in [Efr92, Theorem 1].

We suspect that the expression for  $E_n^*(\tau, s)$  above can be used to show that  $g(\tau)$  is automorphic and that  $\log(g(\tau))$  is a harmonic function on  $\mathfrak{H}^n$  (i.e., is annihilated by invariant differential operators), as is done for  $n = 2$  (see, e.g., [Sie80, § I.2] for automorphy) and  $n = 3$  (see [Efr92, §3]). This may be the approach to answer the question raised at the end of §1 and §3 in [Ter73]. However, we shall not pursue these issues in the present article.

The classical Kronecker limit formula for Eisenstein series can be used to give the polar and constant terms of the Laurent series expansion of the zeta function of a quadratic imaginary field (due to Kronecker) and of a real quadratic field (due to Hecke). Similarly, our generalization of the Kronecker limit formula for Eisenstein series can be used to give the polar and constant terms of the Laurent series expansion of the zeta function of any number field. We briefly indicate how that can be done (this was already done for cubic fields in [Efr92]).

Take  $n$  to be the degree of the number field (over  $\mathbf{Q}$ ). Let  $r$  denote the number of real embeddings and  $s$  denote the number of complex conjugate embeddings. We assume that  $r + s > 1$  since the cases  $r + s = 1$  are already done. The main idea that is used to obtain the polar and constant terms of the Laurent series expansion of the zeta function is to use a generalization of a trick of Hecke to express the zeta function as an integral of the Eisenstein series over a suitable region. The procedure is described for  $n = 3$  in [Efr92, §4] and the discussion there generalizes in an obvious way except for the generalization of the the second-last formula on page 183 of loc. cit. (or analogously the last equation in §4 of loc. cit.). We now indicate how the latter generalization can be achieved.

First consider the case where all embeddings are complex. Let  $m = s - 1$ . Then for any real numbers  $a_1, \dots, a_{m+1}$ , we have

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty (a_1^2 t_1^2 + \cdots + a_m^2 t_m^2 + a_{m+1}^2 (t_1 \cdots t_m)^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\ &= (a_1^2 \cdots a_{m+1}^2)^{-s} \int_0^\infty \cdots \int_0^\infty (t_1^2 + \cdots + t_m^2 + (t_1 \cdots t_m)^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} . \end{aligned}$$

Now consider the case where the field has at least one real embedding. Let  $m = r + s - 1$ . For  $i = 1, \dots, m + 1$ , let  $\delta_i = 1$  if the  $i$ -th embedding is real and  $\delta_i = 2$  otherwise. Without loss of generality, assume that the  $(m + 1)$ -st embedding is real, i.e.,  $\delta_{m+1} = 1$ . Then for any positive real numbers  $a_1, \dots, a_{m+1}$ , we have

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty (a_1^2 t_1^2 + \cdots + a_m^2 t_m^2 + a_{m+1}^2 (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\ &= (a_1^{\delta_1} \cdots a_{m+1}^{\delta_{m+1}})^{-s} \int_0^\infty \cdots \int_0^\infty (t_1^2 + \cdots + t_m^2 + (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} . \end{aligned}$$

The two equations above (depending on what case one is in) provide the desired generalization of the the second-last formula on page 183 of loc. cit.

## 2 Proof of Theorem 1.1

We first prove the formula for  $E_n^*(\tau, s)$  and deduce from it the formula for  $E'_n(\tau, s)$ . In formula (2), the term corresponding to  $m_1 = 0$  is

$$\begin{aligned}
S_1 &= \pi^{-ns/2} \Gamma(ns/2) \sum_{m_2, \dots, m_n} \frac{(\prod_{i=1}^{n-1} y_i^{n-i})^s}{\| (m_2 \dots m_n) \tau' \|^{ns/2}} \\
&= \pi^{-ns/2} \Gamma(ns/2) \cdot \left( \prod_{i=1}^{n-1} y_i^{\binom{i}{n-1}} \right)^s \cdot \sum_{m_2, \dots, m_n} \frac{(\prod_{i=1}^{n-2} y_i^{n-1-i})^{\binom{n}{n-1}s}}{\| (m_2 \dots m_n) \tau' \|^{(n-1)(\frac{n}{n-1}s)/2}} \\
&= \left( \prod_{i=1}^{n-1} y_i^{\binom{i}{n-1}} \right)^s \cdot \pi^{-ns/2} \Gamma(ns/2) \cdot E_{n-1} \left( \tau', \frac{n}{n-1}s \right) \\
&= \left( \prod_{i=1}^{n-1} y_i^{\binom{i}{n-1}} \right)^s \cdot \pi^{-(n-1)(\frac{n}{n-1}s)/2} \Gamma \left( (n-1) \left( \frac{n}{n-1}s \right) / 2 \right) \cdot E_{n-1} \left( \tau', \frac{n}{n-1}s \right) \\
&= \left( \prod_{i=1}^{n-1} y_i^{\binom{i}{n-1}} \right)^s \cdot E_{n-1}^* \left( \tau', \frac{n}{n-1}s \right). \tag{3}
\end{aligned}$$

Let

$$S_2 = \sum_{m_1 \neq 0} \sum_{m_2, \dots, m_n} \frac{\pi^{-ns/2} \Gamma(ns/2)}{\| (m_1 \dots m_n) \tau \|^{ns/2}},$$

so that

$$E_n^*(\tau, s) = S_1 + \left( \prod_{i=1}^{n-1} y_i^{n-i} \right)^s \cdot S_2. \tag{4}$$

Our next goal is to find a suitable expression for  $S_2$ , which will not be achieved till equation (15) below. We use the formula

$$\frac{\pi^{-s} \Gamma(s)}{a^s} = \int_0^\infty \exp(-\pi at) t^s \frac{dt}{t} \tag{5}$$

with  $a = \| (m_1 \dots m_n) \tau \|$ , and  $s$  replaced by  $ns/2$  to get

$$S_2 = \sum_{m_1 \neq 0} \sum_{m_2, \dots, m_n} \int_0^\infty \exp(-\pi t \| (m_1 \dots m_n) \tau \|) t^{ns/2} \frac{dt}{t} \tag{6}$$

Now

$$\begin{aligned}
\| (m_1 \dots m_n) \tau \| &= (m_1 \tau_{1,1})^2 + \dots + \left( \sum_{i=1, \dots, n-1} m_i \tau_{i, n-1} \right)^2 + \left( \sum_{i=1, \dots, n-1} m_i \tau_{i, n} + m_n \tau_{n, n} \right)^2 \\
&= a_n + (b_n + c_n m_n)^2, \tag{7}
\end{aligned}$$

where

$$a_n = (m_1 \tau_{1,1})^2 + \dots + \left( \sum_{i=1, \dots, n-1} m_i \tau_{i, n-1} \right)^2, \tag{8}$$

$b_n = \sum_{i=1, \dots, n-1} m_i \tau_{i,n}$ , and  $c_n = \tau_{n,n}$ . Putting (7) in (6), we get

$$\begin{aligned} S_2 &= \sum_{m_1 \neq 0} \sum_{m_2, \dots, m_n} \int_0^\infty \exp(-\pi t a_n) \exp(-\pi t (b_n + c_n m_n)^2) t^{ns/2} \frac{dt}{t} \\ &= \sum_{m_1 \neq 0} \sum_{m_2, \dots, m_{n-1}} \int_0^\infty \exp(-\pi t a_n) \sum_{m_n} \exp(-\pi t (b_n + c_n m_n)^2) t^{ns/2} \frac{dt}{t} \end{aligned} \quad (9)$$

The Poisson summation formula says that for real numbers  $t, b, c$ , with  $c \neq 0$ ,

$$\sum_{m \in \mathbf{Z}} \exp(-\pi t (b + cm)^2) = \frac{1}{c\sqrt{t}} \sum_{m \in \mathbf{Z}} \exp(2\pi i b m / c) \exp(-\pi m^2 / t c^2). \quad (10)$$

Using this with  $m = m_n$ ,  $b = b_n$ , and  $c = c_n$ , and noting that  $c_n$ , being a diagonal entry of  $\tau$ , is always positive, we get

$$\sum_{m_n} \exp(-\pi t (b_n + c_n m_n)^2) = \frac{1}{c_n \sqrt{t}} \exp(-(\pi t a_n)) \sum_{m_n} \exp(2\pi i b_n m_n / c_n) \exp(-\pi m_n^2 / t c_n^2).$$

Putting this in (9), we get

$$\begin{aligned} S_2 &= \frac{1}{c_n} \sum_{m_1 \neq 0} \sum_{m_2, \dots, m_{n-1}} \int_0^\infty \exp(-\pi t a_n) \sum_{m_n} \exp(2\pi i b_n m_n / c_n) \exp(-\pi m_n^2 / t c_n^2) t^{ns/2-1/2} \frac{dt}{t} \\ &= \frac{1}{c_n} \sum_{m_1 \neq 0} \sum_{m_2, \dots, m_n} \exp(2\pi i b_n m_n / c_n) \int_0^\infty \exp(-(\pi a_n t + \pi m_n^2 / t c_n^2)) t^{ns/2-1/2} \frac{dt}{t} \\ &= \frac{1}{c_n} \sum_{m_1 \neq 0, m_n} \exp(2\pi i b_n m_n / c_n) \\ &\quad \cdot \int_0^\infty \exp(-\pi (m_n^2 / c_n^2) / t) \sum_{m_2, \dots, m_{n-2}} \sum_{m_{n-1}} \exp(-(\pi a_n t)) t^{ns/2-1/2} \frac{dt}{t} \end{aligned} \quad (11)$$

Now from equation (8),

$$\begin{aligned} a_n &= (m_1 \tau_{1,1})^2 + \dots + \left( \sum_{i=1, \dots, n-2} m_i \tau_{i, n-2} \right)^2 + \left( \sum_{i=1, \dots, n-2} m_i \tau_{i, n-1} + m_{n-1} \tau_{n-1, n-1} \right)^2 \\ &= a_{n-1} + (b_{n-1} + c_{n-1} m_n)^2, \end{aligned}$$

where  $a_{n-1} = (m_1 \tau_{1,1})^2 + \dots + (\sum_{i=1, \dots, n-2} m_i \tau_{i, n-2})^2$ ,  $b_{n-1} = \sum_{i=1, \dots, n-2} m_i \tau_{i, n-1}$ , and  $c_{n-1} = \tau_{n-1, n-1}$ . So using formula (10), noting that  $c_{n-1}$ , being a diagonal entry of  $\tau$ , is always positive, we have

$$\begin{aligned} &\sum_{m_{n-1}} \exp(-(\pi a_n t)) \\ &= \exp(-(\pi t a_{n-1})) \sum_{m_{n-1}} \exp(-\pi t (b_{n-1} + c_{n-1} m_{n-1})^2) \\ &= \frac{1}{c_{n-1} \sqrt{t}} \exp(-(\pi t a_{n-1})) \sum_{m_{n-1}} \exp(2\pi i b_{n-1} m_{n-1} / c_{n-1}) \exp(-\pi m_{n-1}^2 / t c_{n-1}^2). \end{aligned}$$

Putting this in (11),

$$\begin{aligned}
S_2 &= \frac{1}{c_{n-1}c_n} \sum_{m_1 \neq 0, m_n} \exp(2\pi i b_n m_n / c_n) \int_0^\infty \exp(-\pi m_n^2 / t c_n^2) \sum_{m_2, \dots, m_{n-2}} \frac{1}{\sqrt{t}} \exp(-(\pi t a_{n-1})) \\
&\quad \cdot \sum_{m_{n-1}} \exp(2\pi i b_{n-1} m_{n-1} / c_{n-1}) \exp(-\pi m_{n-1}^2 / t c_{n-1}^2) t^{ns/2-2/2} \frac{dt}{t} \\
&= \frac{1}{c_{n-1}c_n} \sum_{m_1 \neq 0, m_n} \sum_{m_{n-1}} \exp(2\pi i (b_n m_n / c_n + b_{n-1} m_{n-1} / c_{n-1})) \\
&\quad \cdot \int_0^\infty \exp(-\pi (m_n^2 / c_n^2 + m_{n-1}^2 / c_{n-1}^2) / t) \sum_{m_2, \dots, m_{n-2}} \exp(-(\pi t a_{n-1})) t^{ns/2-2/2} \frac{dt}{t} \\
&= \frac{1}{c_{n-1}c_n} \sum_{m_1 \neq 0, m_n, m_{n-1}} \exp(2\pi i (b_n m_n / c_n + b_{n-1} m_{n-1} / c_{n-1})) \\
&\quad \cdot \int_0^\infty \exp(-\pi (m_n^2 / c_n^2 + m_{n-1}^2 / c_{n-1}^2) / t) \sum_{m_2, \dots, m_{n-3}} \sum_{m_{n-2}} \exp(-(\pi t a_{n-1})) t^{ns/2-2/2} \frac{dt}{t} \quad (12)
\end{aligned}$$

Looking at equations (11) and (12), we see that repeated use of Poisson summation gives

$$S_2 = \frac{1}{\prod_{i=2}^n c_i} \sum_{m_1 \neq 0, m_n, m_{n-1}, \dots, m_2} \exp(2\pi i d) \int_0^\infty \exp(-\pi m^2 / t) \exp(-(\pi t a_1)) t^{ns/2-(n-1)/2} \frac{dt}{t}.$$

Now  $a_1 = m_1^2 y^2$  where  $y = \tau_{1,1} = y_1 y_2 \cdots y_{n-1}$ . So

$$\left( \prod_{i=2}^n c_i \right) S_2 = \sum_{m_1 \neq 0} \sum_{m_2, \dots, m_n} \exp(2\pi i d) \int_0^\infty \exp(-(\pi (m_1 y)^2 t + \pi m^2 / t)) t^{n(s-1)/2+1/2} \frac{dt}{t}.$$

If  $m_2 = \cdots = m_n = 0$ , then the corresponding term becomes

$$S'_2 = \sum_{m_1 \neq 0} \int_0^\infty \exp(-\pi (m_1 y)^2 t) t^{n(s-1)/2+1/2} \frac{dt}{t},$$

which, using formula (5), with  $s$  replaced by  $n(s-1)/2 + 1/2$  becomes

$$\begin{aligned}
S'_2 &= \sum_{m_1 \neq 0} \frac{\pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2)}{(m_1 y)^{2(n(s-1)/2+1/2)}} \\
&= y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2) \cdot 2 \sum_{m_1 \geq 0} \frac{1}{m_1^{n(s-1)+1}} \\
&= 2y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2) \zeta(n(s-1) + 1) \quad (13)
\end{aligned}$$

Let

$$\begin{aligned}
S''_2 &= \left( \prod_{i=2}^n c_i \right) S_2 - S'_2 \\
&= \sum_{m_1 \neq 0} \sum_{(m_2, \dots, m_n) \neq (0, \dots, 0)} \exp(2\pi i d) \int_0^\infty \exp(-(\pi (m_1 y)^2 t + \pi m^2 / t)) t^{n(s-1)/2+1/2} \frac{dt}{t}. \quad (14)
\end{aligned}$$

For  $a$  and  $b$  positive real numbers, recall the function

$$K_s(a, b) = \int_0^\infty \exp(-(a^2 t + b^2/t)) t^s \frac{dt}{t}$$

Noting that  $m \neq 0$  if not all  $m_2, \dots, m_n$  are zero,

$$S_2'' = \sum_{m_1 \neq 0} \sum_{(m_2, \dots, m_n) \neq (0, \dots, 0)} \exp(2\pi i d) K_{n(s-1)/2+1/2}(\sqrt{\pi}|m_1|y, \sqrt{\pi}|m|) \quad (15)$$

From equations (3), (4), (13), (14), and (15), we finally get an expression for  $E_n^*(\tau, s)$ :

$$\begin{aligned} E_n^*(\tau, s) &= \left( \prod_{i=1}^{n-1} y_i^{\left(\frac{i}{n-1}\right)} \right)^s \cdot E_{n-1}^*\left(\tau', \frac{n}{n-1}s\right) \\ &+ 2 \left( \prod_{i=1}^{n-1} y_i^{n-i} \right)^s \frac{1}{\prod_{i=2}^n c_i} y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2 + 1/2) \zeta(n(s-1) + 1) \\ &+ \left( \prod_{i=1}^{n-1} y_i^{n-i} \right)^s \frac{1}{\prod_{i=2}^n c_i} \sum_{m_1 \neq 0} \sum_{\substack{(m_2, \dots, m_n) \\ \neq (0, \dots, 0)}} \exp(2\pi i d) K_{n(s-1)/2+1/2}(\sqrt{\pi}|m_1|y, \sqrt{\pi}|m|) \end{aligned} \quad (16)$$

The good thing about the formula above is that it is easy to read off the polar part and the constant term in each of the summands above, which is what we do now. It is known that  $K_s$  is an entire function of  $s$ , and so all the functions appearing in the expression above are holomorphic at  $s = 1$  except  $\zeta(2s - 1)$ , which has a simple pole at  $s = 1$ , and perhaps  $E_{n-1}^*\left(\tau', \frac{n}{n-1}s\right)$ . By induction,  $E_{n-1}^*\left(\tau', \frac{n}{n-1}s\right)$  is also holomorphic except perhaps when  $\left(\frac{n}{n-1}\right)s = 1$ , and in particular is holomorphic at  $s = 1$ . So the first and last summands on the right side of equation (16) are holomorphic at  $s = 1$ ; using the fact that  $K_{1/2}(a, b) = \frac{\sqrt{\pi}}{a} \exp(-2ab)$ , their sum is

$$\left( \prod_{i=1}^{n-1} y_i^{\left(\frac{i}{n-1}\right)} \right) E_{n-1}^*\left(\tau', \frac{n}{n-1}\right) + \sum_{m_1 \neq 0} \sum_{(m_2, \dots, m_n) \neq (0, \dots, 0)} \exp(2\pi i d) \frac{1}{|m_1|} \exp(-2\pi|m_1||m|y) + O(s-1),$$

which is  $-4 \log g(\tau) + O(s - 1)$ .

In order to deal with the second summand on the right side of equation (16), note that

$$\zeta(n(s-1) + 1) = \frac{1}{n(s-1)} + \gamma + O(s-1),$$

$$\Gamma(n(s-1)/2 + 1/2) = \sqrt{\pi} \left(1 + \frac{n}{2}(\gamma - \log 4)(s-1) + O(s-1)^2\right),$$

$$\pi^{-(n(s-1)/2+1/2)} = \frac{1}{\sqrt{\pi}} \left(1 - \frac{n}{2} \log \pi(s-1) + O(s-1)^2\right),$$

$$y^{-(n(s-1)+1)} = y^{-1} (1 - n \log y(s-1) + O(s-1)^2),$$

and

$$\left( \prod_{i=1}^{n-1} y_i^{n-i} \right)^s = \left( \prod_{i=1}^{n-1} y_i^{n-i} \right)^{(s-1)+1} = \left( \prod_{i=1}^{n-1} y_i^{n-i} \right) (1 + \log \left( \prod_{i=1}^{n-1} y_i^{n-i} \right) (s-1) + O(s-1)^2).$$

Using the formulas above, the second summand in on the right side of equation (16) becomes

$$\frac{2/n}{s-1} + \left( \gamma - \log 4\pi - \frac{2}{n} \log \left( \prod_{i=1}^{n-1} y_i^i \right) \right) + O(s-1)$$

Using the formulas obtained above for the three summands on the right side of equation (16), we get the desired formula for  $E_n^*(\tau, s)$ .

Now

$$\begin{aligned} E_n'(\tau, s) &= \pi^s \Gamma(s)^{-1} E_n^*(\tau, s), \\ \pi^s &= \pi(1 + \log \pi(s-1) + O(s-1)^2), \text{ and} \\ \Gamma(s)^{-1} &= (1 + \gamma(s-1) + O(s-1)^2). \end{aligned}$$

Using the equations above and the formula for  $E_n^*(\tau, s)$  gives the desired formula for  $E_n'(\tau, s)$ .

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